A higher-order deformed Heisenberg spin equation as an exactly solvable dynamical equation

1989 J. Phys. A: Math. Gen. 22 L53

(http://iopscience.iop.org/0305-4470/22/2/002)

J. Phys. A: Math. Gen. 22 (1989) L53–L54. Printed in the UK

LETTER TO THE EDITOR

A higher-order deformed Heisenberg spin equation as an exactly solvable dynamical equation

De-gang Zhang and Jie Liu

Institute of Solid State Physics, Sichuan Normal University, Chengdu 610066, People's Republic of China.

Received 23 September 1988

Abstract. We find that a deformed continuous Heisenberg spin equation is geometrically equivalent to the Hirota equation, for which there exists an N-envelope-soliton solution.

In recent years wide interest has been focused on the close connection between various non-linear systems (Lakshmanan 1977, Porsezian *et al* 1987, Lamb 1976 and Kundu 1984). In this letter, we shall prove that the higher-order deformed Heisenberg spin equation

$$
\begin{aligned}
&\boldsymbol{S}_t = \boldsymbol{S}\times\boldsymbol{S}_{xx} - 3\alpha(\boldsymbol{S}_x\cdot\boldsymbol{S}_x)\boldsymbol{S}_x - 6\alpha(\boldsymbol{S}_x\cdot\boldsymbol{S}_{xx})\boldsymbol{S} - 2\alpha\boldsymbol{S}_{xxx} \\
&\boldsymbol{S}\cdot\boldsymbol{S} = 1 \qquad \boldsymbol{S} = (S_1, S_2, S_3)
\end{aligned}
\tag{1}
$$

where α is a real positive constant, is geometrically equivalent to the Hirota equation (Hirota 1973).

We map (1) on a moving helical space curve described by the orthogonal trihedral $\boldsymbol{e}_i$ $(i=1,2,3)$ which satisfies the Serret–Frenet equations (Lamb 1976)

$$
\begin{aligned}
\boldsymbol{e}_{1x} &= k\boldsymbol{e}_2 \\
\boldsymbol{e}_{2x} &= -k\boldsymbol{e}_1 + \tau\boldsymbol{e}_3 \\
\boldsymbol{e}_{3x} &= -\tau\boldsymbol{e}_2
\end{aligned}
\tag{2}
$$

where the curvature is given by $k=(\boldsymbol{e}_{1x}\cdot\boldsymbol{e}_{1x})^{1/2}$ and the torsion is given by $\tau = k^{-2}\boldsymbol{e}_1\cdot(\boldsymbol{e}_{1x}\times\boldsymbol{e}_{1xx})$.

Putting $\boldsymbol{e}_1=\boldsymbol{S}$ and using (1) and (2), we can obtain $\boldsymbol{e}_{it}$ as follows:

$$
\boldsymbol{e}_{it} = \boldsymbol{\Omega}\times\boldsymbol{e}_i \qquad \boldsymbol{\Omega} = \sum_{i=1}^{3}\Omega_i\boldsymbol{e}_i \tag{3}
$$

where

$$
\begin{aligned}
\Omega_1 &= -\tau^2 + k^{-1}k_{xx} - \alpha k^2\tau - 6\alpha k^{-1}k_{xx}\tau + 2\alpha\tau^3 - 6\alpha k^{-1}k_x\tau_x - 2\alpha\tau_{xx} \\
\Omega_2 &= -k_x + 4\alpha k_x\tau + 2\alpha k\tau_x \\
\Omega_3 &= -k\tau - \alpha k^3 - 2\alpha k_{xx} + 2\alpha k\tau^2.
\end{aligned}
$$

The compatibility of (3) gives rise to the following evolution equations:

$$
\begin{aligned}
k_t &= -2k_x\tau - k\tau_x - 3\alpha k^2k_x - 2\alpha k_{xxx} + 6\alpha k_x\tau^2 + 6\alpha k\tau\tau_x \\
\tau_t &= (-\tau^2 + \tfrac{1}{2}k^2 + k^{-1}k_{xx} - 3\alpha k^2\tau - 6\alpha k^{-1}k_{xx}\tau + 2\alpha\tau^3 - 6\alpha k^{-1}k_x\tau_x - 2\alpha\tau_{xx})_x.
\end{aligned}
\tag{4}
$$

0305-4470/89/020053+02$02.50

One can consider the complex transformation (Lakshmanan 1977)

$$\psi = k(x, t) \exp\left(\mathrm{i} \int_{-\infty}^{x} \tau(x', t)\, \mathrm{d}x'\right). \tag{5}$$

Equations (4) then become the Hirota equation

$$\mathrm{i}\psi_t + \psi_{xx} + \tfrac{1}{2}|\psi|^2\psi + \mathrm{i}3\alpha|\psi|^2\psi_x + \mathrm{i}2\alpha\psi_{xxx} = 0 \tag{6}$$

which is the equivalent form of (1). As the Hirota equation has been solved by the bilinear method (Hirota 1973), we immediately conclude that our higher-order deformed Heisenberg spin equation (1) is exactly solved.

References

Hirota R 1973 *J. Math. Phys.* **14** 805
Kundu A 1984 *J. Math. Phys.* **25** 3433
Lakshmanan M 1977 *Phys. Lett.* **61A** 53
Lamb G Jr 1976 *Phys. Rev. Lett.* **37** 235
Mikhailov A V and Shabat A B 1986 *Phys. Lett.* **116A** 191
Porsezian K, Tamizhmani K M and Lakshmanan M 1987 *Phys. Lett.* **124A** 159